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Stellarator symmetry

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Abstract

A simple and general definition of stellarator symmetry is presented and its relation to previous definitions discussed. It is shown that the field-line flow in systems possessing stellarator symmetry is time-reversal invariant if the the toroidal angle is regarded as "time".

Keywords: Time reversal; Fusion; Magnetic field; Toroidal confinement; Stellarator symmetry

1. Introduction

The possession of a reversing symmetry is a simplifying feature of many dynamical systems [1,2]. For example, a Hamiltonian, $H(q, p, t)$, which is an even function of the momentum p and time t generates a dynamics which is invariant under time reversal $t \mapsto -t$ combined with the phase-space involution $q \mapsto q$, $p \mapsto -p$. Knowledge that a system has such a symmetry is useful when trying to locate invariant sets since it implies that $p = 0$ is a symmetry plane – invariant sets are symmetric about the symmetry plane, or at least occur in pairs that are symmetric about it. For example if only a single pair of elliptic and hyperbolic periodic orbits of a given rotation number survives a small perturbation of an integrable Hamiltonian system (the typical case) then they must both be symmetric. In a $1\frac{1}{2}$-dimensional Hamiltonian system, by the Poincaré–Birkhoff theorem [3], the elliptic and hyperbolic orbits interlace, so if the system possesses the above symmetry, then one of these two orbits must have a point on the symmetry line, which makes the search for periodic orbits much easier [4].

In most Hamiltonian systems in physics, time-reversal invariance is obvious and exact. However, in the case of magnetic field-line flow in toroidal magnetic plasma confinement devices, which is known to be a Hamiltonian system [6, pp. 170–175], the "time" is a toroidal angle, and in a non-axisymmetric system it is not obvious a priori that the dynamics is time-reversal invariant.

Such non-axisymmetric systems occur in the class of magnetic confinement devices known as stellarators, and all modern stellarators are designed (and, to within engineering tolerances, constructed,) to have a certain symmetry known as "stellarator symmetry". The Large Helical Device at the National Institute for Fusion Science, Japan, and Wendelstein 7X at the Max-Planck-Institut für Plasmaphysik, Germany, are very large experiments under construction, while the 4-field period heliac TJ-II at CIEMAT, Spain, has just

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Fig. 1. Schematic of the coil set for the H-1 heliac, showing how it produces a 3-fold symmetric magnetic field, illustrated by lines on a magnetic surface (invariant torus). This configuration possesses the property of stellarator symmetry defined in the text.

commenced operation and the 3-field period heliac H-1 at The Australian National University (see Fig. 1) has been running for some years [7].

Because these devices are inherently three-dimensional (in contrast to the axisymmetry of an ideal tokamak) their design poses considerable conceptual and computational challenges. It is thus important to find and exploit any symmetries that they possess. In this note we define stellarator symmetry in its simplest and most general form and show that it may be regarded as a time-reversal symmetry of the magnetic field line flow. Its relation to a symmetry of curvilinear coordinates, also called stellarator symmetry, is discussed.

2. Stellarator symmetry

Consider a vector field $\boldsymbol{F}(\boldsymbol{r})$, where $\boldsymbol{r}$ is the position vector. (The examples considered in this paper are the current density $\boldsymbol{J}$ and magnetic field $\boldsymbol{B}$). Representing the position vector in cylindrical coordinates,

$$\boldsymbol{r} = \rho \boldsymbol{e}_\rho(\phi) + z\boldsymbol{e}_z, \tag{1}$$

where we use the orthonormal basis $\boldsymbol{e}_\rho \equiv \nabla\rho, \boldsymbol{e}_\phi \equiv \rho\nabla\phi, \boldsymbol{e}_z \equiv \nabla z$, we define the symmetry operation

$$I_0 f(\rho, \phi, z) \equiv f(\rho, -\phi, -z) \tag{2}$$

for an arbitrary function $f(\rho, \phi, z)$. This operation is a cylindrical inversion symmetry about the half line $\{\phi = 0, z = 0, \rho > 0\}$.

Then we say $\boldsymbol{F}$ possesses *stellarator symmetry* if there exists a cylindrical coordinate system (ρ, ϕ, z) such that the following symmetry obtains for $\rho > 0$, $\phi \in [0, 2\pi)$ and $z \in \mathbb{R}$

$$I_0[F_\rho, F_\phi, F_z] = [-F_\rho, F_\phi, F_z], \tag{3}$$

where F_ρ, F_ϕ and F_z are the components of $\boldsymbol{F}$ with respect to the orthonormal basis defined above.

Lemma 1. If stellarator symmetry exists with respect to the cylindrical inversion operation I_0, defined about the half line $\{\phi = 0, z = 0\}$, then it also exists with respect to the operation I_π defined about the half line $\{\phi = \pi, z = 0\}$.

This follows by making the substitution $\phi \mapsto \phi + \pi$ in Eq. (3) and invoking the 2π-periodicity of physical quantities.

Stellarators are typically composed of a number of theoretically identical sectors, called field periods, giving rise to a discrete rotational symmetry of the system. Thus, if there is a symmetry half-line in one field period there must be a corresponding symmetry half-line in all field periods:

Lemma 2. If stellarator symmetry exists with respect to I_0 for a vector field possessing N-fold discrete symmetry about the z-axis, then stellarator symmetry also exists with respect to the symmetry operations I_{2n} defined as cylindrical inversions in the half lines $\{\phi = 2n\pi/N, z = 0\}$, where $n = 1, 2, \ldots, (N-1)$.

This follows directly from the definition of N-fold symmetry as applied to the half line in the definition of stellarator symmetry, $\{\phi = 0, z = 0\}$.

Applying the discrete symmetry to the half line $\{\phi = \pi, z = 0\}$ of Lemma 1 we see also that, if

N is odd, stellarator symmetry also applies about a symmetry half-line in the middle of each field period:

Lemma 3. If stellarator symmetry exists with respect to I_0 for a vector field possessing N-fold discrete symmetry about the z-axis, where N is an odd integer, then stellarator symmetry also exists with respect to the symmetry operations I_{2n-1} defined as cylindrical inversions in the half lines $\{\phi = (2n-1)\pi/N, z = 0\}$, where $n = 1, 2, \ldots, N$.

(In the case of N an even integer, $\phi = \pi$ is equivalent to $\phi = 0$ and Lemma 1 is redundant.)

Suppose a spatial distribution of current density $\boldsymbol{J}$ (including both coil and plasma currents) has stellarator symmetry, i.e. Eq. (3) applies with $\boldsymbol{F} = \boldsymbol{J}$. Writing out Ampère's law $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ in cylindrical coordinates

$$\left[\left(\frac{1}{\rho}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right), \left(\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho}\right), \frac{1}{\rho}\left(\frac{(\rho B_\phi)}{\partial \rho} - \frac{\partial B_\rho}{\partial \phi}\right)\right] = \mu_0 [J_\rho, J_\phi, J_z], \tag{4}$$

and observing that $[\partial_\rho, \partial_\phi, \partial_z] I_0 f = I_0 [\partial_\rho, -\partial_\phi, -\partial_z] f$, we see that Eq. (4) is consistent with the magnetic field $\boldsymbol{B}$ *also* satisfying stellarator symmetry. Thus (assuming no magnetic field at infinity) it follows from the uniqueness of $\boldsymbol{B}$ that:

Lemma 4. A magnetic field possesses stellarator symmetry if and only if it is produced by a current distribution with stellarator symmetry.

3. Field-line flow

For the purposes of this section we define field-line flow by the equation of motion $\dot{\boldsymbol{r}} = \rho \boldsymbol{B}/B_\phi$, where $\boldsymbol{r}$ is the position vector, whose time rate of change is the "velocity" $\dot{\boldsymbol{r}} \equiv \dot{\rho}\boldsymbol{e}_\rho + \rho\dot{\phi}\boldsymbol{e}_\phi + \dot{z}\boldsymbol{e}_z$. From the ϕ-component of the equation of motion we recognize that $\dot{\phi} = 1$, so "time" t is identified with the toroidal angle ϕ. The remaining two components form the two-dimensional dynamical system

$$\dot{\rho} = \rho \frac{B_\rho}{B_\phi}, \tag{5}$$

$$\dot{z} = \rho \frac{B_z}{B_\phi}. \tag{6}$$

From conservation of magnetic flux we see that Eqs. (5) and (6) conserve the measure $B_\phi \, \mathrm{d}\rho \, \mathrm{d}z$. In fact [6, pp. 170–175], as discussed in Section 5, field-line flow can be represented as a Hamiltonian dynamical system, in which case the measure preserving property becomes the standard area-preservation under time-evolution in $1\frac{1}{2}$-dimensional Hamiltonian systems. In the present section it suffices to work in non-canonical form.

From Eq. (3) (with $\boldsymbol{F} = \boldsymbol{B}$) we see that the dynamical system defined by Eqs. (5) and (6) is *time-reversal invariant under the phase-space involution* $\rho \mapsto \rho$, $z \mapsto -z$. It follows that if $\{\rho(\phi), z(\phi)\}$ is a field line then so also is $\{\rho(-\phi), -z(-\phi)\}$ (though not necessarily the same line). The intersection of an invariant set with the half plane $\{\phi = 0\}$ is either symmetric about the half line $\{\phi = 0, z = 0\}$ or is one member of a pair that is symmetric, thus facilitating the search for periodic orbits (closed field lines) just as in the Hamiltonian and area-preserving map cases [4].

A recent practical application [5] is illustrated in Fig. 2 which shows a Poincaré section in the symmetry plane $\{\phi = 0\}$ of the field line flow in the H-1 heliac at the Australian National University, a stellarator possessing stellarator symmetry. The "up-down" symmetry of the flow is apparent and two symmetric periodic orbits are in evidence – the elliptic and hyperbolic periodic orbits associated with the $(5, 3)$ island chain. Both periodic orbits always intersect the symmetry line $\{\phi = 0, \ z = 0\}$, which makes their location easy. Three cases are shown, each with slightly different vertical field coil currents, showing an interchange of stability between the two periodic orbits in the two outer cases and the virtual disappearence of the island in the intermediate case.

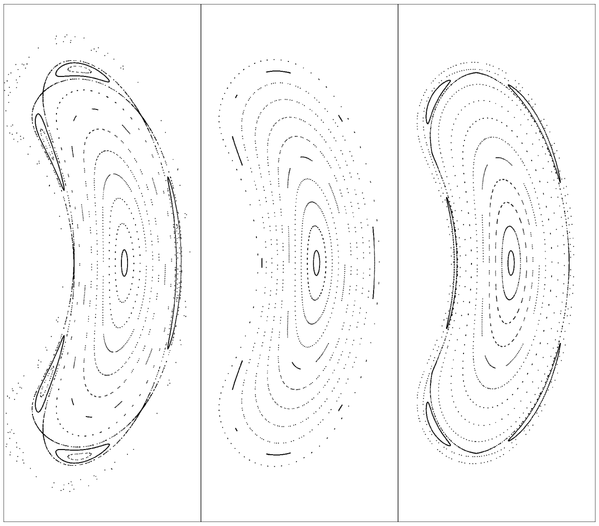

Fig. 2. Manipulation of the (5, 3) magnetic island in the vacuum field of H-1. On the left is the standard configuration. The middle and right plots are obtained by variation of the currents in the vertical field coils.

4. Flux coordinates

One of the main uses of stellarator symmetry has been in simplifying the representation of the mapping from a curvilinear coordinate system (s, θ, ζ) based on the magnetic field to real space cylindrical coordinates (ρ, ϕ, z):

$$\begin{aligned}\rho &= R(s, \theta, \zeta),\\ z &= Z(s, \theta, \zeta),\\ \phi &= \Phi(s, \theta, \zeta).\end{aligned} \tag{7}$$

Here s is coordinate labelling a continuous set of nested tori, θ a poloidal angle coordinate increasing by 2π on one circuit the short way around a torus $s =$ const, and ζ is a toroidal angle increasing by 2π the long way round. The existence of the symmetry allows the Fourier representation of R, Z and Φ to be simplified by enabling the use of only a sine or a cosine basis, $\sin(m\theta - n\zeta)$ or $\cos(m\theta - n\zeta)$, depending on the parity of the quantity in question under the transformation to be defined below.

The toroidal angle ζ can be taken to be the geometric angle ϕ [8,9], but often a more general angle is used [6], whose level surfaces are not planar. A particularly popular choice is to use "Boozer coordinates" [10,11]. Usually the coordinate s is chosen so that the tori $s =$ const are invariant under the magnetic field line flow if possible. That is, $\boldsymbol{B} \cdot \nabla s$ should either vanish, if an invariant torus of the desired rotation number exists, or else, the coordinate system should be constructed so as to make $\boldsymbol{B} \cdot \nabla s$ small in some suitable sense [12,13].

Sometimes [8] stellarator symmetry is stated in terms of the "up-down" symmetry illustrated in Fig. 2 – the existence of a plane, $\zeta \equiv \phi = 0$, such that $R(s, \theta, 0) = R(s, -\theta, 0)$, $Z(s, \theta, 0) = -Z(s, -\theta, 0)$. However, as we have seen above, stellarator symmetry must involve the full ϕ-domain. A more satisfactory definition is $R(s, -\theta, -\phi) = R(s, \theta, \phi)$, $Z(s, -\theta, -\phi) = -Z(s, \theta, \phi)$ [9]. Generalizing the choice of toroidal angle ζ, define the symmetry operation

$$S_0 g(s, \theta, \zeta) \equiv g(s, -\theta, -\zeta) \tag{8}$$

for any function $g(s, \theta, \zeta)$. Then this definition of stellarator symmetry becomes the property

$$S_0 f(R, \Phi, Z) \equiv f(R, -\Phi, -Z). \tag{9}$$

We shall call any curvilinear coordinate system having the symmetry Eq. (9) *stellarator-symmetry coordinates.*

This is a rather different definition from the one put forward in the present paper, Eq. (3), as it is a property of the coordinate system, rather than a property of the physical vector fields. However, it is closely connected by the following:

Lemma 5. When acting on a quantity expressed in stellarator-symmetry coordinates, the operation S_0 is equivalent to the operation I_0 acting on the same quantity expressed in cylindrical coordinates.

We can use this equivalence, for example, to find how stellarator-symmetry coordinates, regarded as functions of position, transform under cylindrical inversion about the half line $\{\phi = 0, z = 0\}$: $I_0[s, \theta, \zeta] = S_0[s, \theta, \zeta] \equiv [s, -\theta, -\zeta]$.

The contravariant representation of stellarator symmetry in stellarator-symmetry coordinates is

$$S_0[F^s, F^\theta, F^\zeta] = [-F^s, F^\theta, F^\zeta], \tag{10}$$

where F^s, F^θ and F^ζ are the contravariant components $\boldsymbol{F}\cdot\nabla s$, $\boldsymbol{F}\cdot\nabla\theta$ and $\boldsymbol{F}\cdot\nabla\zeta$, respectively. This may be verified by using the identity $\boldsymbol{F}\cdot\nabla\eta \equiv F_\rho\partial_\rho\eta + \rho^{-1}F_\phi\partial_\phi\eta + F_z\partial_z\eta$, where η is s, θ or ζ. Then replace S_0 by I_0, using Lemma 5, and verify that the three terms making up $\boldsymbol{F}\cdot\nabla\eta$ are each of odd parity under I_0.

Note that, if the torus $s = s_0 =$ const is invariant under the magnetic field line flow, then $B^s(s_0,\theta,\zeta)\equiv 0$. Thus the s-component of Eq. (10) applied to $\boldsymbol{B}$ is trivially satisfied, showing that it is compatible with stellarator symmetry to base a coordinate system on invariant tori (in so far as they exist).

We end this section by giving the covariant analogue of Eq. (10)

$$S_0[F_s, F_\theta, F_\zeta] = [-F_s, F_\theta, F_\zeta], \tag{11}$$

where F_s, F_θ and F_ζ are the covariant components $\boldsymbol{F}\cdot\partial_s\boldsymbol{r}$, $\boldsymbol{F}\cdot\partial_\theta\boldsymbol{r}$ and $\boldsymbol{F}\cdot\partial_\zeta\boldsymbol{r}$, respectively. To show this, use the identity $\boldsymbol{F}\cdot\partial_\eta\boldsymbol{r} \equiv F_\rho\partial_\eta R + RF_\phi\partial_\eta\Phi + F_z\partial_\eta Z$, where η is s, θ or ζ. Then Eq. (11) is easily verified by checking the parity of each term.

5. Field-line Hamiltonian

We now make the special choice $s = \psi$, where $2\pi\psi$ is the toroidal flux function such that [6,12]

$$\boldsymbol{B}(\boldsymbol{r}) = \nabla\psi\times\nabla\theta + \nabla\zeta\times\nabla\chi, \tag{12}$$

Dotting Eq. (12) with $\nabla\theta$ and $\nabla\psi$ we see that $\partial_\psi\chi = \mathcal{J}B^\theta$ and $\partial_\theta\chi = -\mathcal{J}B^\psi$, where $\mathcal{J}\equiv 1/\nabla\psi\cdot\nabla\theta\times\nabla\zeta$ is the Jacobian of the transformation from Cartesian to curvilinear coordinates. It is readily verified that $\mathcal{J}$ is even under I_0 or S_0. Thus, from Eq. (11), if $\boldsymbol{B}$ possesses stellararator symmetry then $\partial_\psi\chi$ is even under S_0, while $\partial_\theta\chi$ is odd, whence we deduce that χ is even.

The magnetic field line flow is equivalent to a time-dependent $1\frac{1}{2}$-degree-of-freedom Hamiltonian system [12]

$$\dot\theta = \partial_\psi\chi, \tag{13}$$

$$\dot\psi = -\partial_\theta\chi. \tag{14}$$

where the dot represents the derivative with respect to ζ. These equations are recognized as Hamilton's equations with ζ acting like a time coordinate, ψ playing the role of the momentum conjugate to the angular position coordinate, θ, and $\chi\equiv\chi(\psi,\theta,\zeta)$ being the Hamiltonian.

Since χ is even under S_0, we see that Eqs. (13) and (14) are invariant under S_0. Since ζ is "time", they are *time-reversal invariant under the phase-space involution* $\psi\mapsto\psi$, $\theta\mapsto-\theta$. This is in contrast to the usual case in Hamiltonian mechanics when it is the momentum variable that changes sign in the involution.

6. Conclusion

We have shown that there is a simple time-reversal symmetry for magnetic field-line flows possessing stellarator symmetry. Unlike most cases of time-reversal symmetry, stellarator symmetry is an artefact of Man rather than a Law of Nature. Thus it can never be exact and an interesting question is whether breaking of stellarator symmetry (by design, due to fabrication errors or due to plasma instabilities) destroys time-reversal invariance or just makes the corresponding phase-space involution more complicated.

Another possible area for further investigation is the application of the theory of k-symmetry [14] to stellarators due to their N-fold discrete symmetry.

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